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Technical Report 544

M. H. Hayes

The Uniqueness of a Multi-dimensional Sequence in Terms of Its Phase or Magnitude

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FOR THE COMMANDER

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THE UNIQUENESS OF A MULTI-DIMENSIONAL SEQUENCE IN TERMS OF ITS PHASE OR MAGNITUDE

M. H. HAYES
Group 27

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ABSTRACT

A multi-dimensional sequence is not, in general, uniquely defined in terms of only the phase or magnitude of its Fourier transform. However, in this paper some conditions are developed under which a multi-dimensional sequence is uniquely defined by its phase. A similar set of conditions are then developed for the unique specification of a multi-dimensional sequence in terms of its Fourier transform magnitude. In both cases, it is initially assumed that either the phase or magnitude is known for all frequencies. The results are then generalized to the case in which the phase or magnitude is known only for a finite set of frequency values.

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I. INTRODUCTION

Under a variety of conditions, a multi-dimensional (m-D) sequence may be reconstructed from partial information about its Fourier transform. For example, if an m-D sequence $x(n_1,\ldots,n_m)$ is zero whenever any of the indices n_k for $k=1,\ldots,m$ are negative, then $x(n_1,\ldots,n_m)$ can be exactly recovered from the real part or, except for $x(0,\ldots,0)$, from the imaginary part of its Fourier transform [1]. If, on the other hand, $x(n_1,\ldots,n_m)$ is minimum phase [2], then it can be recovered from the magnitude [illegible] to within a scale factor, from the phase of its Fourier transform.

The reconstruction of an m-D sequence from such partial information is important and useful in many applications. For example, in some cases of optical image processing or in the measurement of diffraction patterns, only spectral magnitude information can be recorded or is available. Therefore, it is of interest to recover a signal from only spectral magnitude information in these cases. In other situations, either the spectral magnitude or phase of a signal may be severely distorted so that the restoration of the signal must rely on the undistorted component. For example, in the class of problems referred to as blind deconvolution [3], a signal is to be recovered from an observation which is the convolution of a desired signal with some unknown signal. Since little is usually known about either the desired signal or the distorting signal, deconvolution of the two signals is generally a very difficult problem. However, in the special case in which the distorting signal is known to have a Fourier transform which is purely real, the tangent of the phase of the observed signal is identical to the tangent of the phase of the original signal. Such a situation occurs, at least approximately, in long-term exposure to atmospheric turbulence or when images are blurred by severely defocused lenses with circular aperture stops [4]. In this case, it is of interest to consider signal reconstruction from phase information alone.

This paper is concerned with the development of some conditions under which an m-D sequence is uniquely defined by its Fourier transform phase or magnitude. In general, of course, phase or magnitude information alone is not sufficient to uniquely specify an m-D sequence. For example, any m-D sequence may be convolved with a zero phase sequence to produce another m-D sequence with the same phase. Similiarly, any m-D sequence may be convolved with an all-pass sequence to produce another m-D sequence with the same magnitude. Therefore, without any additional information or constraints, the Fourier transform phase or magnitude may, at best, uniquely specify an m-D sequence to within a zero phase or all-pass convolutional factor, respectively. Nevertheless, with a few basic results from the theory of polynomials in several variables, some useful conditions may be derived under which an m-D sequence is uniquely defined by the phase or magnitude of its Fourier transform. These conditions, which are distinctly different from the minimum or maximum phase constraints, imply that most m-D sequences with finite support are recoverable from either their phase or magnitude.

This paper is organized as follows. In Section II, the necessary results from the algebra of polynomials in more than one variable are briefly reviewed. Some notation and terminology related to multi-dimensional signals is then presented in Section III. In Section IV, conditions are developed under which an m-D sequence is uniquely defined in terms of the phase or magnitude of its

Fourier transform. Although these conditions are initially derived under the assumption that either the phase or magnitude is known for all frequencies, they are then extended to the case in which the phase or magnitude is known only over a finite set (lattice) of points.

II. POLYNOMIALS IN MORE THAN ONE VARIABLE

In this section, some notation and terminology related to the algebra of polynomials in more than one variable is reviewed. In addition, two theorems are presented which are of considerable importance in many multi-dimensional signal processing applications and will be referred to frequently in this paper. Proofs of these theorems as well as a detailed treatment of many topics not presented in this section may be found in [5].

A monomial is a function of n variables of the form:

$$f = c\, z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} \tag{1}$$

where $k_1, k_2, \ldots, k_n$ are non-negative integers and c is an arbitrary number which is referred to as the coefficient of the monomial. The degree of the monomial is defined as

$$d(f) = k_1 + k_2 + \ldots + k_n \tag{2}$$

A polynomial is then simply a sum of a finite number of monomials. In this paper, a polynomial will be referred to as being non-trivial if it consists of a sum of two or more monomials. A trivial polynomial is therefore either a constant or a monomial of non-zero degree.

The coefficients of the monomials which define a polynomial are called the coefficients of the polynomial and the degree of the polynomial is the degree of the monomial with the highest degree. Any polynomial p in n variables of degree N may therefore be written in the form:

$$p = \sum_{k_1+\ldots+k_n \leq N} \cdots \sum c(k_1,\ldots,k_n)\, z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} \tag{3}$$

It is often useful to consider p in (3) as a polynomial in one variable, say z_ℓ, with coefficients which are polynomials in the remaining (n-1) variables. For example, p in (3) may be written as

$$p = \sum_{k=0}^{N} \phi_\ell(k)\, z_\ell^k \tag{4}$$

where $\phi_\ell(k)$ for $k=0,1,\ldots,N$ are polynomials in the (n-1) variables z_n for $n \neq \ell$. In this form, the largest value of k for which $\phi_\ell(k)$ is *non-zero* is referred to as the degree of p with respect to the variable z_ℓ and will be denoted by $d_\ell(p)$.

If all the coefficients of a polynomial p belong to a particular number field, $\mathscr{F}$, then p is called a polynomial over the field $\mathscr{F}$. The set of all polynomials in n variables over a field $\mathscr{F}$ form a ring which is denoted by $\mathscr{F}(z_1,\ldots,z_n)$. If two polynomials p_1 and p_2 in $\mathscr{F}(z_1,\ldots,z_n)$ are such that $p_1=cp_1$ where $c \in \mathscr{F}$ and is non-zero, then p_1 and p_2 are called associated polynomials. A polynomial $p \in \mathscr{F}(z_1,\ldots,z_n)$ with $d(p) > 0$ is called a reducible polynomial over the field $\mathscr{F}$ if there exists polynomials $p_1,p_2 \in \mathscr{F}(z_1,\ldots,z_n)$ such that $p=p_1p_2$ with $d(p_1) > 0$ and $d(p_2) > 0$. If no such decomposition is possible, then p is called an irreducible polynomial. It is of interest to note that, as a consequence of the Fundamental Theorem of Algebra, the only polynomials in one variable over the field of complex numbers which are irreducible are polynomials of first degree. This result, however, is not true in general. Over the field of real numbers, for example, the polynomial $p(z)=z^2+1$ is irreducible. Even over the field of complex numbers, the polynomial $p(z_1,z_2)=z_1^2+z_2$ is irreducible. This second example illustrates, in particular, the fact that the Fundamental Theorem of Algebra does not hold for polynomials in more than one variable.

The first theorem of interest in this paper asserts that any polynomial of non-zero degree can be uniquely decomposed, to within factors of zero degree, into a product of irreducible polynomials. More specifically,

> <u>Theorem 1</u>: Any polynomial $p \in \mathscr{F}(z_1,\ldots,z_n)$ having non-zero degree can be expressed as a product of factors irreducible in $\mathscr{F}$. Furthermore, if p has two different factorizations:
>
> $$p = f_1\, f_2 \ldots f_m = g_1\, g_2 \ldots g_n \tag{5}$$
>
> then m=n and the factors f_i and g_i can be ordered in such a way that the factors are associated.

It is well-known [5] that a polynomial p(z) in one variable of degree N is uniquely defined in terms of its values over a set $A=\left\{a_k\right\}_{k=0}^{N}$ of N+1 distinct points. This result has a natural extension to polynomials in more than one variable which asserts the uniqueness of a polynomial $p \in \mathscr{F}(z_1,\ldots,z_n)$ in terms of its values over an n-dimensional lattice of points. An n-dimensional lattice is an extension of a single set of points, A, to an n-fold Cartesian product of n sets of points, A_k, for k=1,...,n. Specifically, let $A_k = \{a_{k,\ell}\}_{\ell=1}^{N_k}$ be a set of N_k distinct points in the field $\mathscr{F}$ for k=1,...,n. Then the n-dimensional lattice $\mathscr{L}(A_1,\ldots,A_n)$ is defined as

$$\mathscr{L}(A_1,\ldots,A_n) = \prod_{k=1}^{n} A_k = A_1 \times A_2 \times \ldots \times A_n \tag{6}$$

The result of interest may now be stated as follows:

<u>Theorem 2</u>: Suppose $p_1, p_2 \in \mathscr{F}(z_1,\ldots,z_n)$ with $d_k(p_1) < N_k$ and $d_k(p_2) < N_k$ for k=1,...,n. Let A_k be a set of N_k distinct points in the field $\mathscr{F}$. If p_1 and p_2 are equal over the set (lattice) of points $\mathscr{L}(A_1,\ldots,A_n)$, then $p_1=p_2$.

This theorem will be used in deriving some conditions for the uniqueness of a multi-dimensional sequence in terms of a finite set (lattice) of values of either the phase or magnitude of its Fourier transform.

III. FRAMEWORK

This paper is concerned with the uniqueness of a multi-dimensional sequence with real coefficients in terms of either the phase or magnitude of its Fourier transform. Although the results which are presented apply to sequences of arbitrary dimension, all discussions are phrased in terms of two-dimensional sequences, $x(n_1,n_2)$, in order to simplify notation. In this section, some notation and terminology related to 2-D sequences is presented and the general framework of the uniqueness problem is established.

The z-transform of a two-dimensional sequence, $x(n_1,n_2)$ is defined by

$$X(z_1,z_2) = \sum_{n_1}\sum_{n_2} x(n_1,n_2)\, z_1^{n_1}\, z_2^{n_2} \tag{7}$$

and the Fourier transform, denoted by $X(\omega_1,\omega_2)$, is equal to $X(z_1,z_2)$ on the unit bi-disc $|z_1| = |z_2| = 1$, i.e.,

$$X(\omega_1,\omega_2) = X(z_1,z_2)\Big|_{\substack{z_1=\exp(j\omega_1)\\ z_2=\exp(j\omega_2)}} \tag{8}$$

Written in polar form, $X(\omega_1,\omega_2)$ is represented in terms of its magnitude and phase as

$$X(\omega_1,\omega_2) = |X(\omega_1,\omega_2)|\exp[j\phi_x(\omega_1,\omega_2)] \tag{9}$$

where it is assumed that the phase is defined by its principal value. Therefore, in terms of (9), this paper considers the uniqueness of $x(n_1,n_2)$ in terms of $\phi_x(\omega_1,\omega_2)$ or $|X(\omega_1,\omega_2)|$. Although initially it is assumed that either the phase or magnitude is known for all values of ω_1 and ω_2, the results are then extended to the case in which the phase or magnitude is known only over a finite set of points.

Most of the sequences which are considered in this paper have finite support, i.e., $x(n_1,n_2)$ is non-zero only for finitely many values of its arguments n_1 and n_2. A sequence with finite support is said to be of extent $N_1 x N_2$ if $x(n_1,n_2)=0$ outside a rectangular region of the form $[K_0,K_0+N_1-1] \times [L_0,L_0+N_2-1]$ for some K_0 and L_0. For the special case in which K_0 and L_0 may be taken to be equal to zero, $x(n_1,n_2)$ is said to have first quadrant support and the region of support is denoted by $\mathscr{R}(N_1,N_2)$, i.e.,

$$\mathscr{R}(N_1,N_2) = [0, N_1-1] \times [0, N_2-1] \qquad (10)$$

Throughout this paper, any sequence with finite support may be assumed, without any loss in generality, to have first quadrant support. In the general case, a sequence may simply be shifted in order to satisfy this assumption. Consequently, the set $F(n_1,n_2)$ will be used to denote the collection of all real 2-D sequences of finite extent with first quadrant support. The notation $x \varepsilon F(n_1,n_2)$ will therefore mean that the sequence $x(n_1,n_2)$ has support $\mathscr{R}(N_1,N_2)$ for some N_1 and N_2. Since the z-transform of a sequence $x \varepsilon F(n_1,n_2)$ is a polynomial in two variables, z_1 and z_2, $X(z_1,z_2)$ is an element of $\mathscr{F}(z_1,z_2)$ over the field of real numbers:

$$x \varepsilon F(n_1,n_2) \longleftrightarrow X \varepsilon \mathscr{F}(z_1,z_2) \qquad (11)$$

Now, suppose that $x \varepsilon F(n_1,n_2)$ and has an irreducible z-transform of degree N_1 in z_1 and N_2 in z_2 and consider the sequence $\tilde{x} \varepsilon F(n_1,n_2)$ defined by

$$\tilde{x}(n_1,n_2) = x(N_1-n_1, N_2-n_2) \tag{12}$$

The z-transform of $\tilde{x}(n_1,n_2)$ is also irreducible and is given by

$$\tilde{X}(z_1,z_2) = z_1^{N_1} z_2^{N_2} X(z_1^{-1},z_2^{-1}) \tag{13}$$

A special class of sequences which will play an important role in the following sections consists of those sequences for which

$$x(n_1,n_2) = \pm\tilde{x}(n_1,n_2) \tag{14}$$

or,

$$X(z_1,z_2) = \pm\tilde{X}(z_1,z_2) \tag{15}$$

Since $\tilde{x}(n_1,n_2)$ corresponds to a 180^o rotation of $x(n_1,n_2)$, sequences which satisfy (14) are, except possibly for a minus sign, invariant under 180^o rotations. Therefore, these sequences will be said to be symmetric or to have symmetric z-transforms. It should be noted that a 1-D sequence which has a symmetric z-transform has all of its zeros (excluding those at $z=0$ or $z^{-1}=0$) on the unit circle or in reciprocal pairs. Therefore, (14) represents an extension of this property to 2-D sequences. It may also be noted that except for a linear phase term of the form $\exp[\frac{1}{2}j(N_1\omega_1+N_2\omega_2)]$, the Fourier transform of a symmetric sequence is either purely real or purely imaginary.

An important property of a sequence $x \in F(n_1,n_2)$ is that its z-transform need only be known over a finite set of points in order to uniquely specify the sequence. Although these points can not be chosen arbitrarily, Theorem 2 in Section II provides one set of points which is sufficient for this specification. More specifically, the following lemma is a direct consequence of Theorem 2:

<u>Lemma</u>: Suppose $x, y \in F(n_1,n_2)$ with support $\mathscr{R}(N_1,N_2)$. Let $A_k = \{a_{k,\ell}\}_{\ell=1}^{M_k}$ be a set of M_k distinct complex numbers for k=1,2 with $M_1 > N_1$ and $M_2 > N_2$. If

$$X(z_1,z_2)\Big|_{\mathscr{L}(A_1,A_2)} = Y(z_1,z_2)\Big|_{\mathscr{L}(A_1,A_2)} \tag{16}$$

then $x(n_1,n_2) = y(n_1,n_2)$ for all n_1 and n_2.

Of particular interest in this paper, however, is the case in which the elements of the sets A_k have unit magnitude. Specifically, consider the sets

$$\Omega_k = \left\{\beta_{k,\ell}\right\}_{\ell=1}^{M_k} \quad \text{with} \quad 0 \leq \beta_{k,\ell} < 2\pi \tag{17a}$$

$$A_k = \left\{\exp(\, j\beta_{k,\ell}\,)\right\}_{\ell=1}^{M_k} \tag{17b}$$

where the elements of Ω_k for k=1,2 are assumed to be distinct. Then

$$X(z_1,z_2)\Big|_{\mathscr{L}(A_1,A_2)} = X(\omega_1,\omega_2)\Big|_{\mathscr{L}(\Omega_1,\Omega_2)} \tag{18}$$

represents the Fourier transform of $x(n_1,n_2)$ evaluated over the lattice $\mathscr{L}(\Omega_1,\Omega_2)$ in the ω_1,ω_2-plane. If, in addition, the numbers $\beta_{k,\ell}$ are equally spaced between 0 and 2π, i.e., $\beta_{k,\ell}=2\pi\ell/M_k$, then (18) represents the $M_1 x M_2$-point Discrete Fourier Transform (DFT) of $x(n_1,n_2)$. In this case, the $M_1 x M_2$-point DFT will be denoted by $X(k_1,k_2)_{M_1,M_2}$:

$$X(k_1,k_2)_{M_1,M_2} = X(z_1,z_2)\Big|_{\substack{z_1=\exp(j2\pi k_1/M_1)\\ z_2=\exp(j2\pi k_2/M_2)}} \tag{19}$$

which, when expressed in terms of its magnitude and phase, will be written as

$$X(k_1,k_2)_{M_1,M_2} = |X(k_1,k_2)|_{M_1,M_2} \exp[j\phi_x(k_1,k_2)_{M_1,M_2}] \tag{20}$$

In the following sections, when the length of the DFT is assumed to be known or when it is explicitly stated, the subscripts in (20) will occasionally be dropped.

IV. THE QUESTION OF UNIQUENESS

In this section, the results from Section II are applied to the question of the uniqueness of a 2-D sequence in terms of the phase or magnitude of its Fourier transform. In Section 4.1, conditions are given under which a 2-D sequence with finite support is uniquely specified in terms of the phase of its Fourier transform. A similiar set of conditions are given in Section 4.2 for the uniqueness of a 2-D sequence with finite support in terms of its Fourier transform magnitude. In Section 4.3, the results in Sections 4.1 and 4.2 are used to generate a set of conditions under which a 2-D sequence with finite support in uniquely specified in terms of either its Fourier transform phase or magnitude. In addition, the generalization of these results to 2-D sequences whose convolutional inverses have finite support is described.

4.1 Uniqueness in terms of phase

It has recently been shown [6] that a 1-D finite length sequence $x(n)$ is uniquely specified to within a scale factor by the phase or the tangent of the phase of its Fourier transform if $X(z)$ has no zeros on the unit circle or in reciprocal pairs, i.e., if $X(z)$ contains no symmetric factors. This result may be directly extended to the case of multi-dimensional sequences. This extension, in terms of 2-D sequences is as follows:

<u>Theorem 3</u>: Let $x,y \in F(n_1,n_2)$. If $X(z_1,z_2)$ and $Y(z_1,z_2)$ have no symmetric factors and

$$\phi_x(\omega_1,\omega_2) = \phi_y(\omega_1,\omega_2) \tag{21a}$$

for all ω_1 and ω_2, then $y(n_1,n_2) = \beta\, x(n_1,n_2)$ for some positive number β. If, on the other hand,

$$\tan \phi_x(\omega_1,\omega_2) = \tan \phi_y(\omega_1,\omega_2) \tag{21b}$$

for all ω_1 and ω_2, then $y(n_1,n_2) = \beta\, x(n_1,n_2)$ for some real number β.

It should be noted that the symmetric factors which are excluded from $X(z_1,z_2)$ and $Y(z_1,z_2)$ in this theorem need not be irreducible. For example, if $A(z_1,z_2)$ is a polynomial and $X(z_1,z_2)=P(z_1,z_2)Q(z_1,z_2)$ where $P(z_1,z_2)=A(z_1,z_2)\tilde{A}(z_1,z_2)$, then $x(n_1,n_2)$ does not satisfy the constraints of the theorem since $P(z_1,z_2)$ is a (reducible) symmetric factor of $X(z_1,z_2)$. In effect, the exclusion of symmetric factors from a sequence $x \in F(n_1,n_2)$ is equivalent to the constraint that if $A(z_1,z_2)$ is an irreducible factor of $X(z_1,z_2)$ then $\tilde{A}(z_1,z_2)$ is not a factor of $X(z_1,z_2)$.

An outline of a proof of Theorem 3 is as follows. Let $x,y \in F(n_1,n_2)$ and let N be a positive integer which is sufficiently large so that $x(n_1,n_2)$ and $y(n_1,n_2)$ are both zero outside the domain $\mathscr{R}(N,N)$. Consider the sequence

$$h(n_1,n_2) = x(n_1,n_2) * y(-n_1,-n_2) \tag{22}$$

which has a z-transform given by

$$H(z_1,z_2) = X(z_1,z_2)\, Y(z_1^{-1},z_2^{-1}) \tag{23}$$

By noting that the phase of $h(n_1,n_2)$ is given by

$$\phi_h(\omega_1,\omega_2) = \phi_x(\omega_1,\omega_2) - \phi_y(\omega_1,\omega_2) \tag{24}$$

it follows from (21a) that $\phi_h(\omega_1,\omega_2)=0$ or from (21b) and the trigonometric identity

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \tag{25}$$

that $\tan\phi_h(\omega_1,\omega_2) = 0$. In either case, the Fourier transform of $h(n_1,n_2)$ is purely real which implies that

$$h(n_1,n_2) = h(-n_1,-n_2) \tag{26}$$

Therefore, from (23) and (26),

$$X(z_1,z_2)\ Y(z_1^{-1},z_2^{-1}) = X(z_1^{-1},z_2^{-1})\ Y(z_1,z_2) \tag{27}$$

Multiplying both sides of (27) by $(z_1z_2)^N$ results in the following polynomial equation:

$$X(z_1,z_2)\ \tilde{Y}(z_1,z_2)\ z_1^{k_1}\ z_2^{k_2} = \tilde{X}(z_1,z_2)\ Y(z_1,z_2)\ z_1^{\ell_1}\ z_2^{\ell_2} \tag{28}$$

where k_1,k_2,ℓ_1, and ℓ_2 are non-negative integers. Now consider an arbitrary non-trivial irreducible factor $X_k(z_1,z_2)$ of $X(z_1,z_2)$. From Theorem 1 in

Section 2, it follows that $X_k(z_1,z_2)$ must be associated either with a factor of $\tilde{X}(z_1,z_2)$ or with a factor of $Y(z_1,z_2)$. However, if $X_k(z_1,z_2)$ is associated with a factor of $\tilde{X}_k(z_1,z_2)$ then

$$X_k(z_1,z_2) = \alpha \tilde{X}_\ell(z_1,z_2) \tag{29}$$

for some ℓ. If $\ell=k$ then (29) implies that

$$X_k(z_1,z_2) = \alpha^2 X_k(z_1,z_2) \tag{30}$$

Therefore, $\alpha = \pm 1$ and $X_k(z_1,z_2)$ is symmetric. If, on the other hand, $k \neq \ell$ then

$$X_k(z_1,z_2)\, X_\ell(z_1,z_2) = \alpha \tilde{X}_\ell(z_1,z_2)\, X_\ell(z_1,z_2) \tag{31}$$

In other words, both $X_\ell(z_1,z_2)$ and $\tilde{X}_\ell(z_1,z_2)$ are factors of $X(z_1,z_2)$. Both cases, however, are excluded by the theorem hypothesis. Consequently, each non-trivial irreducible factor of $X(z_1,z_2)$ must be associated with a factor of $Y(z_1,z_2)$. Therefore, $Y(z_1,z_2)$ is of the form:

$$Y(z_1,z_2) = \left[z_1^{m_1}\, z_2^{m_2}\, P(z_1,z_2) \right] X(z_1,z_2) \tag{32}$$

where m_1 and m_2 are integers and $P(z_1,z_2)$ is a polynomial. However, as in the steps leading to (26), (21) requires that the term in brackets correspond to a sequence with a real Fourier transform. This, in turn, implies that $P(z_1,z_2)$

is symmetric which, since $Y(z_1,z_2)$ contains no symmetric factors, requires that $P(z_1,z_2)$ be a constant:

$$Y(z_1,z_2) = \beta z_1^{m_1} z_2^{m_2} X(z_1,z_2) \tag{33}$$

Again using (21) it follows that $m_1=m_2=0$ and the desired result follows by noting that β must be positive in the case of hypothesis (21a).

It may be apparent at this point that it is not necessary to assume that (21) holds for all ω_1 and ω_2 in order to prove the theorem if the regions of support of $x(n_1,n_2)$ and $y(n_1,n_2)$ are known. More specifically, suppose that $x(n_1,n_2)$ and $y(n_1,n_2)$ are known to be zero outside the domain $\mathscr{R}(N_1,N_2)$. If $M_1 > 2(N_1-1)$ and $M_2 > 2(N_2-1)$ and if (21) is replaced with the constraint that the phase or tangent of the phase of the $M_1 x M_2$-point DFT's of $x(n_1,n_2)$ and $y(n_1,n_2)$ are equal, then Theorem 3 still remains valid. This follows first from the observation that $h(n_1,n_2)$ in (22) is zero outside the region $[(1-N_1),(N_1-1)] x [(1-N_2),(N_2-1)]$. Therefore, the $M_1 x M_2$-point DFT of $h(n_1,n_2)$ equals the product of the $M_1 x M_2$-point DFT's of $x(n_1,n_2)$ and $y(n_1,n_2)$. Thus,

$$\phi_h(k_1,k_2)_{M_1,M_2} = \phi_x(k_1,k_2)_{M_1,M_2} - \phi_y(k_1,k_2)_{M_1,M_2} \tag{34}$$

and (27) follows as in the proof of Theorem 3. Proceeding as in Theorem 3, it follows that $Y(z_1,z_2)$ and $X(z_1,z_2)$ are related by (32). However, the constraints on the phase of the $M_1 x M_2$-point DFT's of $x(n_1,n_2)$ and $y(n_1,n_2)$ imply that the term in brackets in (32) corresponds to a sequence with a real $M_1 x M_2$-point DFT. Therefore, in a style similiar to that in Theorem 3, it follows that the term in brackets in a constant. Therefore a corollary to Theorem 3 is as follows:

Corollary 3.1: Let $x,y \in F(n_1,n_2)$ with support $\mathscr{R}(N_1,N_2)$. If $X(z_1,z_2)$ and $Y(z_1,z_2)$ have no symmetric factors and

$$\phi_y(k_1,k_2)_{M_1,M_2} = \phi_x(k_1,k_2)_{M_1,M_2} \tag{35}$$

with $M_1 > 2(N_1-1)$ and $M_2 > 2(N_2-1)$, then $y(n_1,n_2) = \beta\, x(n_1,n_2)$ for some positive number β. If, on the other hand,

$$\tan \phi_y(k_1,k_2)_{M_1,M_2} = \tan \phi_x(k_1,k_2)_{M_1,M_2} \tag{36}$$

then $y(n_1,n_2) = \beta\, x(n_1,n_2)$ for some real number β.

The importance of this corollary lies in the fact that it allows for the development of practical algorithms for reconstructing a sequence from the phase of its DFT.

4.2 Uniqueness in terms of magnitude

In Section 4.1, the uniqueness of a multi-dimensional sequence in terms of the phase of its Fourier transform was considered. This section addresses the dual problem related to the uniqueness of a multi-dimensional sequence in terms of its Fourier transform magnitude. Perhaps the first treatment of this question of uniqueness may be found in [7] where the uniqueness of a 2-D sequence $x \in F(n_1,n_2)$ is stated to be related to the irreducibility of its z-transform. In this section, a slightly more general result is derived which includes sequences with irreducible z-transforms as a special case. Even more importantly, however, as in section 4.1, the uniqueness of a *sequence* $x \in F(n_1,n_2)$ is also considered when the magnitude of its Fourier transform is known only over a finite set (lattice) of points.

Consider a sequence $x \in F(n_1,n_2)$ for which $|X(\omega_1,\omega_2)|$ is known for all ω_1 and ω_2. Since the inverse Fourier transform of $|X(\omega_1,\omega_2)|^2$ is the auto-correlation, $r_x(n_1,n_2)$, of $x(n_1,n_2)$:

$$r_x(n_1,n_2) = x(n_1,n_2) * x(-n_1,-n_2) \tag{37}$$

the specification of $|X(\omega_1,\omega_2)|$ is equivalent to the knowledge of $r_x(n_1,n_2)$ or its z-transform:

$$R_x(z_1,z_2) = X(z_1,z_2)\, X(z_1^{-1},z_2^{-1}) \tag{38}$$

For any $x \in F(n_1,n_2)$, the most general form for its z-transform, $X(z_1,z_2)$ is given by

$$X(z_1,z_2) = \beta\, z_1^{m_1}\, z_2^{m_2} \prod_{k=1}^{p} X_k(z_1,z_2) \tag{39}$$

where β is a real number, m_1 and m_2 are non-negative integers, and $X_k(z_1,z_2)$

for k=1,...,p are non-trivial irreducible polynomials. Substituting (39) into (38) gives

$$R_x(z_1,z_2) = \beta^2 \prod_{k=1}^{p} X_k(z_1,z_2)\, X_k(z_1^{-1},z_2^{-1}) \qquad (40)$$

Now suppose that the polynomial

$$P(z_1,z_2) = \prod_{k=1}^{p} X_k(z_1,z_2) \qquad (41)$$

is of degree N_1 in z_1 and N_2 in z_2. Multiplying $R_x(z_1,z_2)$ by $z_1^{N_1} z_2^{N_2}$ yields a polynomial in z_1 and z_2 which has degree $2N_1$ in z_1 and $2N_2$ in z_2:

$$Q_x(z_1,z_2) = z_1^{N_1}\, z_2^{N_2}\, R_x(z_1,z_2) = \beta^2 \prod_{k=1}^{p} X_k(z_1,z_2)\, \tilde{X}_k(z_1,z_2) \qquad (42)$$

It is apparent that the polynomials $Q_x(z_1,z_2)$ and $|X(\omega_1,\omega_2)|$ contain exactly the same information about $x(n_1,n_2)$ since one may be uniquely derived from the other. Therefore, the ability to uniquely recover $x(n_1,n_2)$ from $|X(\omega_1,\omega_2)|$ is equivalent to the ability to uniquely recover $X(z_1,z_2)$ from $Q_x(z_1,z_2)$. With this in mind, it follows that $x(n_1,n_2)$ cannot be unambiguously recovered from magnitude information alone. For example, the sign of β as well as the linear phase term $z_1^{m_1} z_2^{m_2}$ is not recoverable from $Q_x(z_1,z_2)$. Even more importantly, however, is the observation that, without additional information, it is not possible to determine whether $X_k(z_1,z_2)$ or $\tilde{X}_k(z_1,z_2)$ is a factor of $X(z_1,z_2)$. This ambiguity is not surprising, however, since it represents a 2-D extension of a familiar result for 1-D sequences [8]. Specifically, for any finite duration sequence $x(n)$, another sequence $y(n)$ may be generated which has the same Fourier transform magnitude as $x(n)$ by simply reflecting a zero of $X(z)$ about the unit circle. For 2-D sequences, $\tilde{X}_k(z_1,z_2)$ represents the reflection of the zero contour of $X_k(z_1,z_2)$ about the unit bi-disc $|z_1| = |z_2| = 1$.

It will be useful in the following discussions to define an equivalence relation on the set $F(n_1,n_2)$ as follows:

$$y(n_1,n_2) \sim x(n_1,n_2) \qquad \text{if} \qquad y(n_1,n_2) = \pm x(k_1 \pm n_1, k_2 \pm n_2) \tag{43}$$

for some integers k_1 and k_2. In other words, the equivalence class generated by a sequence $x(n_1,n_2)$ is defined to be the set of all sequences which may be derived from $x(n_1,n_2)$ by a linear shift, a time-reversal, or by a change in sign of the sequence. Note that all of the sequences within a given equivalence class have the same Fourier transform magnitude. Thus, it will be convenient to refer to the Fourier transform magnitude of the sequences within an equivalence class as the Fourier transform magnitude of the class.

In general, there will be more than one equivalence class having the same Fourier transform magnitude. More specifically, given a sequence $x \in F(n_1,n_2)$ there may exist another sequence $y \in F(n_1,n_2)$ with the same Fourier transform magnitude as $x(n_1,n_2)$ but which is not in the same equivalence class as $x(n_1,n_2)$. Therefore, the goal of this section is to determine a set of conditions which guarantee the existence of only one equivalence class with a given Fourier transform magnitude. The first question to be addressed, however, concerns the number of equivalence classes which have a given Fourier transform magnitude. Once this has been established, conditions which guarantee the existence of only one equivalence class may easily be determined. The answer to this first question is implied by the following theorem:

Theorem 4: Let $x \varepsilon F(n_1,n_2)$ with a z-transform given by

$$X(z_1,z_2) = \beta z_1^{m_1} z_2^{m_2} \prod_{k=1}^{p} X_k(z_1,z_2) \tag{44}$$

where $X_k(z_1,z_2)$ are non-trivial irreducible polynomials for $k=1,\ldots,p$. If $y \varepsilon F(n_1,n_2)$ and

$$|X(\omega_1,\omega_2)| = |Y(\omega_1,\omega_2)| \tag{45}$$

for all ω_1 and ω_2, then $Y(z_1,z_2)$ is of the form:

$$Y(z_1,z_2) = \pm \beta z_1^{\ell_1} z_2^{\ell_2} \prod_{k \varepsilon I} X_k(z_1,z_2) \prod_{k \notin I} \tilde{X}_k(z_1,z_2) \tag{46}$$

where I is a subset of the integers in the interval [1,p].

This theorem is simply a statement of the fact that the only way to generate a new sequence, $y(n_1,n_2)$, which has the same Fourier transform magnitude as $x(n_1,n_2)$ is to convolve $x(n_1,n_2)$ with an all-pass sequence, $h(n_1,n_2)$, i.e., a sequence with $|H(\omega_1,\omega_2)| = 1$ for all ω_1 and ω_2. However, any all-pass sequence with a rational z-transform is of the form:

$$H(z_1,z_2) = \pm z_1^{k_1} z_2^{k_2} \prod_{k=1}^{n} H_k^{-1}(z_1,z_2) \tilde{H}_k(z_1,z_2) \tag{47}$$

Therefore, given a sequence $x \varepsilon F(n_1,n_2)$ with a z-transform of the form (44), $y(n_1,n_2) = x(n_1,n_2) * h(n_1,n_2)$ has finite support if and only if for each k, $H_k(z_1,z_2) = X_\ell(z_1,z_2)$ for some $\ell \varepsilon [1,p]$. Consequently, $Y(z_1,z_2)$ must be of the form given by (46).

An outline of a proof of this theorem is as follows. With $x,y \varepsilon F(n_1,n_2)$, let N be an integer large enough so that $x(n_1,n_2)$ and $y(n_1,n_2)$ are zero outside the domain $\mathscr{R}(N,N)$. From (45) it follows that

$$X(z_1,z_2)\ X(z_1^{-1},z_2^{-1}) = Y(z_1,z_2)\ Y(z_1^{-1},z_2^{-1}) \tag{48}$$

Therefore, let the z-transform of $y(n_1,n_2)$ be given by

$$Y(z_1,z_2) = \alpha z_1^{\ell_1} z_2^{\ell_2} \prod_{k=1}^{q} Y_k(z_1,z_2) \tag{49}$$

where $Y_k(z_1,z_2)$ are non-trivial irreducible factors for $k=1,\ldots,q$. Substituting (44) and (49) into (48) and multiplying by $(z_1z_2)^N$ yields the following equation:

$$\beta^2 z_1^{M_1} z_2^{M_2} \prod_{k=1}^{p} X_k(z_1,z_2)\tilde{X}_k(z_1,z_2) = \alpha^2 z_1^{L_1} z_2^{L_2} \prod_{k=1}^{q} Y_k(z_1,z_2)\tilde{Y}_k(z_1,z_2) \tag{50}$$

where M_1, M_2, L_1, and L_2 are positive integers. From Theorem 1 in Section 2, it follows that $M_1=L_1$, $M_2=L_2$, and p=q:

$$\beta^2 \prod_{k=1}^{p} X_k(z_1,z_2)\, \tilde{X}_k(z_1,z_2) = \alpha^2 \prod_{k=1}^{q} Y_k(z_1,z_2)\, \tilde{Y}_k(z_1,z_2) \tag{51}$$

Again from Theorem 1, it follows that the factors $Y_k(z_1,z_2)$ may be ordered in such a way that $Y_k(z_1,z_2)$ is associated with either $X_k(z_1,z_2)$ or $\tilde{X}_k(z_1,z_2)$ for each k. Therefore, from (49) and the fact that (45) implies $\alpha = \pm\beta$, the desired result (46) follows.

It should be noted that, as a consequence of this theorem, all sequences in $F(n_1,n_2)$ with a given magnitude, $|X(\omega_1,\omega_2)|$, have z-transforms with the same number, p, of non-trivial irreducible factors. Furthermore, except for a scale factor of (-1) and linear shifts, the only way to generate another sequence $y(n_1,n_2)$ in $F(n_1,n_2)$ with the same magnitude as $x(n_1,n_2)$ is to replace one or more non-trivial factors $X_k(z_1,z_2)$ of $X(z_1,z_2)$ with $\tilde{X}_k(z_1,z_2)$. However, if $X_k(z_1,z_2)$ is symmetric, then the replacement of $X_k(z_1,z_2)$ with $\tilde{X}_k(z_1,z_2)$ may only change $X(z_1,z_2)$ by a factor of (-1). Therefore, it follows that the number of equivalence classes with magnitude $|X(\omega_1,\omega_2)|$ is at most $2^{(\bar{p}-1)}$ where $\bar{p}$ is the number of non-symmetric irreducible factors in $X(z_1,z_2)$. Thus, the following is an immediate consequence of Theorem 4:

<u>Theorem 5</u>: Let $x \in F(n_1,n_2)$ have a z-transform with at most one irreducible non-symmetric factor, i.e.

$$X(z_1,z_2) = P(z_1,z_2) \prod_{k=1}^{p} X_k(z_1,z_2) \tag{52}$$

where $X_k(z_1,z_2)$ for $k=1,\ldots,p$ are irreducble symmetric factors. If $y \in F(n_1,n_2)$ with

$$|Y(\omega_1,\omega_2)| = |X(\omega_1,\omega_2)| \tag{53}$$

for all ω_1 and ω_2, then $y(n_1,n_2) \sim x(n_1,n_2)$.

As in Section IV, it may be apparent that the assumption that (53) holds for all ω_1 and ω_2 in this theorem is not necessary if $x(n_1,n_2)$ and $y(n_1,n_2)$ are known to be *zero outside some given domain*. More specifically, suppose that for some N_1 and N_2, $x(n_1,n_2)$ and $y(n_1,n_2)$ are known to be zero outside the domain $\mathscr{R}(N_1,N_2)$. Let Ω_k and A_k be sets of M_k distinct points as defined in (17) for $k=1,2$ and let $\mathscr{L}(\Omega_1,\Omega_2)$ and $\mathscr{L}(A_1,A_2)$ be the 2-D lattices generated by these sets. Note that if

$$|X(\omega_1,\omega_2)|\Big|_{\mathscr{L}(\Omega_1,\Omega_2)} = |Y(\omega_1,\omega_2)|\Big|_{\mathscr{L}(\Omega_1,\Omega_2)} \tag{54}$$

then

$$Q_x(z_1,z_2)\Big|_{\mathscr{L}(A_1,A_2)} = Q_y(z_1,z_2)\Big|_{\mathscr{L}(A_1,A_2)} \tag{55}$$

where $Q_x(z_1,z_2)$ and $Q_y(z_1,z_2)$ are polynomials, as defined by (42), of degree at most $2(N_1-1)$ in z_1 and $2(N_2-1)$ in z_2. Therefore, if $M_1 > 2(N_1-1)$ and $M_2 > 2(N_2-1)$ then it follows from Theorem 2 that $Q_x(z_1,z_2) = Q_y(z_1,z_2)$ for all z_1 and z_2. Thus, (54) implies that (53) holds for all ω_1 and ω_2. Consequently, Corollary 5.1: Let $x, y \varepsilon F(n_1,n_2)$ with support $\mathscr{R}(N_1,N_2)$. If $X(z_1,z_2)$ has at most one irreducible non-symmetric factor and

$$|Y(\omega_1,\omega_2)|\Big|_{\mathscr{L}(\Omega_1,\Omega_2)} = |X(\omega_1,\omega_2)|\Big|_{\mathscr{L}(\Omega_1,\Omega_2)} \tag{56}$$

where Ω_k is a set of M_k distinct real numbers in the interval $[0,2\pi)$ with $M_k > 2(N_k-1)$ for k=1,2, then $y(n_1,n_2) \sim x(n_1,n_2)$.

A special case of this theorem results when the points in the sets Ω_k are equally spaced between 0 and 2π. In this instance,

$$|X(\omega_1,\omega_2)|\Big|_{\mathscr{L}(\Omega_1,\Omega_2)} = |X(k_1,k_2)|\Big|_{M_1,M_2} \tag{57}$$

is the magnitude of the $M_1 x M_2$-point DFT of $x(n_1,n_2)$. Therefore, (56) may be replaced with the constraint:

$$|Y(k_1,k_2)|\Big|_{M_1,M_2} = |X(k_1,k_2)|\Big|_{M_1,M_2} \tag{58}$$

4.3 Extensions

In Section 4.1, a set of conditions are presented under which a 2-D sequence is uniquely defined to within a scale factor by the phase of its Fourier transform. A similiar set of conditions are presented in Section 4.2 which allow a 2-D sequence to be uniquely specified by the magnitude of its Fourier transform to within a delay, a sign, and a 180^{0} rotation. It is of interest to note that these uniqueness constraints are not mutually exclusive. Specifically, if $x \in F(n_1,n_2)$ has a non-symmetric irreducible z-transform, then $x(n_1,n_2)$ satisfies the constraints of Theorems 3 and 5. Therefore, the following result is a direct consequence of these theorems:

> Theorem 6: If $x \in F(n_1,n_2)$ and has a non-symmetric irreducible z-transform, then $x(n_1,n_2)$ is uniquely specified (in the sense of Theorems 3 and 5) by either the phase or magnitude of its Fourier transform. If, in addition, $x(n_1,n_2)$ is known to have support $\mathscr{R}(N_1,N_2)$, then the phase or magnitude of the $M_1 x M_2$-point DFT of $x(n_1,n_2)$ is sufficient for this unique specification provided $M_1 > 2(N_1-1)$ and $M_2 > 2(N_2-1)$.

Since a non-symmetric irreducible z-transform is a sufficient constraint for a sequence $x \in F(n_1,n_2)$ to be uniquely defined by either its magnitude or phase, it is of interest to determine how restrictive this constraint is. In other words, given an arbitrary 2-D sequence, is it likely that its z-transform is non-symmetric and irreducible? An answer to this question may be found by considering the set $\mathscr{P}$ of all polynomials in two variables which have degree $N_1 \geq 1$ in z_1 and $N_2 \geq 1$ in z_2. With $N = (N_1+1)(N_2+1)$, note that there exists a one-to-one correspondence between $\mathscr{P}$ and R^N. Specifically, each polynomial $p \in \mathscr{P}$ may be uniquely represented as a vector $\underline{x}_p \in R^N$. Conversely, each point $\underline{x}_p \in R^N$ corresponds to a polynomial $p \in \mathscr{P}$. Now, consider the subset $\mathscr{B}$ of $\mathscr{P}$

which contains all those polynomials which are reducible. With the correspondence noted above, $\mathscr{B}$ represents a subset $\mathscr{U}$ of R^N. Since it may be shown [9] that $\mathscr{U}$ is a set of measure zero in R^N, "almost every" polynomial in more than one variable is irreducible. In a probabilistic setting, this result states that a polynomial $p \varepsilon \mathscr{P}$ is irreducible with probability one. It may also be shown, in a style similiar to that in [9] , that the set of symmetric polynomials represents a set of measure zero in R^N. Thus, it follows that "almost every" polynomial in two or more variables satisfies the constraints of Theorem 6. Consequently, "almost every" 2-D sequence with finite support is uniquely defined (in the sense of Theorems 3 and 5) by either the phase or magnitude of its Fourier transform.

Although the results which have been presented thus far have been confined to sequences with finite support, an extension is easily made to those sequences whose convolutional inverses have finite support. Specifically, let $x_i(n_1,n_2)$ denote the convolutional inverse of a 2-D sequence $x(n_1,n_2)$, i.e.

$$x(n_1,n_2) * x_i(n_1,n_2) = \delta(n_1,n_2) \tag{59}$$

where $\delta(n_1,n_2)$ is the 2-D unit sample function. Now suppose that $x(n_1,n_2)$ is a stable sequence which has a z-transform of the form

$$X(z_1,z_2) = \frac{1}{P(z_1,z_2)} \tag{60}$$

where $p(z_1,z_2)$ is a polynomial in $\mathscr{F}(z_1,z_2)$. In this case, the convolutional inverse of $x(n_1,n_2)$ has a z-transform given by

$$X_i(z_1,z_2) = P(z_1,z_2) \tag{61}$$

so that $x_i \varepsilon F(n_1,n_2)$. In addition, the phase or magnitude of $x_i(n_1,n_2)$ is uniquely defined by the phase or magnitude of $x(n_1,n_2)$ respectively:

$$|X_i(\omega_1,\omega_2)| = |X(\omega_1,\omega_2)|^{-1} \tag{62a}$$

$$\phi_{x_i}(\omega_1,\omega_2) = -\phi_x(\omega_1,\omega_2) \tag{62b}$$

Therefore, if $x(n_1,n_2)$ is a stable sequence with a z-transform given by (60), then $x(n_1,n_2)$ is uniquely defined by its phase or magnitude if the polynomial $P(z_1,z_2)$ satisfies the appropriate constraints.

V. SUMMARY

In this paper, conditions have been developed under which a multi-dimensional sequence is uniquely specified by the phase or magnitude of its Fourier transform. These conditions were shown to include almost all sequences in two or more unknowns which have finite support. Although it was initially assumed that either the phase or magnitude was known for all frequencies, the uniqueness constraints were then extended to include the case in which the phase or magnitude was specified over a particular finite set of points.

The question of uniqueness, however, is only one aspect of the more general problem of multi-dimensional signal reconstruction from phase or magnitude. Although not considered in this paper, another important question concerns the development of practical algorithms for reconstructing a multi-dimensional sequence from either its phase or magnitude. A number of algorithms have been developed for multi-dimensional signal reconstruction from phase which include the extension of 1-D phase-only reconstruction algorithms [6] to higher dimensional sequences. Reconstruction from magnitude, on the other hand, still remains an area of active research. Although a few algorithms have been proposed (some references may be found in [7]), none appear to be suitable for the reconstruction of arbitrary multi-dimensional sequences. Even beyond reconstruction algorithms, another important issue which remains to be investigated is the sensitivity of a multi-dimensional sequence to measurement errors in its phase or magnitude. In particular, it is of interest to determine the effect on the reconstruction of a sequence to noisy phase or magnitude information.

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A multi-dimensional sequence is not, in general, uniquely defined in terms of only the phase or magnitude of its Fourier transform. However, in this paper some conditions are developed under which a multi-dimensional sequence is uniquely defined by its phase. A similar set of conditions are then developed for the unique specification of a multi-dimensional sequence in terms of its Fourier transform magnitude. In both cases, it is initially assumed that either the phase or magnitude is known for all frequencies. The results are then generalized to the case in which the phase or magnitude is known only for a finite set of frequency values.

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